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On strictly locally homogeneous Riemannian manifolds*

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Dedicated to the memory of Franco Tricerri and his family

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Abstract: A coordinate expression is given for a family of 5-dimensional locally homogeneous Riemannian manifolds which are not locally isometric to any globally homogeneous Riemannian space. All these manifolds have volume-preserving local geodesic symmetries.

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1. Introduction

It is well known that there exist locally homogeneous Riemannian manifolds which are not locally isometric to any globally homogeneous Riemannian manifold (see e.g. [4, 6, 8, 9, 10, 11]). The simplest of such examples appear in dimension 5 and can be described as follows (cf. [4, p. 214]): let G be the matrix group $SO(3) \times SO(3)$ and H_r its subgroup of all matrices of the form

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where t is a variable and $r \in \mathbb{R}$ is a fixed parameter. If r is a rational number, then one can construct the homogeneous space G/H_r and a naturally reductive G-invariant Riemannian metric on it. If r is irrational, then the subgroup H_r is not closed in G and the homogeneous space G/H_r in the usual sense does not exist. Yet, choosing an $\mathrm{ad}(H_r)$-invariant scalar product on the coset space $\mathfrak{g}/\mathfrak{h}_r$ of the corresponding Lie algebras, one can hence define a locally homogeneous Riemannian manifold of dimension 5 which is not locally isometric to a globally homogeneous Riemannian space. To obtain a coordinate expression for such a situation (and more general situations) one can use the following construction whose idea belongs to Lastaria and Tricerri [8]:

Theorem A. *Let G be a Lie group, $\mathfrak{g}$ the corresponding Lie algebra, and let $\mathfrak{g} = V + \mathfrak{h}$ be a decomposition of $\mathfrak{g}$ into a vector subspace and a Lie subalgebra such that $[\mathfrak{h}, V] \subset V$.*

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Further, let a scalar product $\langle\cdot,\cdot\rangle$ *be given on* V *such that the adjoint representation of* $\mathfrak{h}$ *on* V *is skew-symmetric with respect to* $\langle\cdot,\cdot\rangle$. *Let* $\{E_1,\dots,E_n,H_1,\dots,H_r\}$ *be a basis of left-invariant vector fields on* G *such that* $\{E_1,\dots,E_n\}$ *is an orthonormal basis of* V *and* $\{H_1,\dots,H_r\}$ *is a basis of* $\mathfrak{h}$. *Let* $\{\theta^1,\dots,\theta^n,\omega^1,\dots,\omega^r\}$ *be the dual basis of invariant differential* 1*-forms on* G. *Finally, let* $\{x^1,\dots,x^n,x^{n+1},\dots,x^{n+r}\}$ *be a local coordinate system defined in a neighborhood* U *of the identity* $e\in G$ *by a coordinate chart* $\Phi\colon U\to V\subset\mathbb{R}^{n+r}$ *such that* $\Phi(e)=(0,\dots,0,0,\dots,0)$ *and*

$$(dx^i)_e=(\theta^i)_e \text{ for } i=1,\dots,n.$$

Put

$$f(a^1,\dots,a^n)=\Phi^{-1}(a^1,\dots,a^n,0,\dots,0).$$

Then there is an open neighborhood M *of the origin* $o\in\mathbb{R}^n[a^1,\dots,a^n]$ *for which* $f\colon M\to G$ *is an imbedding and the pullbacks* $\tilde\theta^1,\dots,\tilde\theta^n$ *of the* 1*-forms* $\theta^1,\dots,\theta^n$ *via* f *are linearly independent. The domain* M *equipped with the Riemannian metric*

$$g=\sum_{i=1}^{n}\tilde\theta^i\otimes\tilde\theta^i$$

is a locally homogeneous Riemannian manifold. Moreover, the tangent map f_{*o} *gives a linear isometry between* (T_oM,g_o) *and* $(V,\langle\cdot,\cdot\rangle)$.

Corollary. *If, in addition,* G *is equipped with a left-invariant Riemannian metric* $\tilde g$, *the decomposition* $\mathfrak{g}=V+\mathfrak{h}$ *is orthogonal with respect to the induced scalar product in* T_eG *and* $\langle\cdot,\cdot\rangle$ *is the restriction of* $\tilde g_e$ *to* $V\subset T_eG$, *then the corresponding Riemannian metric* g *on* M *is induced by the metric* $\tilde g$ *on* G *via* f.

Proof of Theorem A is a slight modification of the proof of Theorem 4.1 from [8] and will be omitted. We only note that the decomposition of $\mathfrak{g}$ as above with the skew-symmetric adjoint action of $\mathfrak{h}$ on $(V,\langle\cdot,\cdot\rangle)$ is equivalent to prescribing an "infinitesimal model" (T,K) on $(V,\langle\cdot,\cdot\rangle)$ as required in the original theorem (see [8] for more details).

The Corollary is obvious.

Let us mention that if the connected Lie subgroup $H\subset G$ corresponding to the Lie subalgebra $\mathfrak{h}$ is topologically closed in G, then the locally homogeneous manifold constructed in Theorem A is locally isometric to the homogeneous space G/H equipped with the G-invariant Riemannian metric corresponding to the $\mathrm{ad}(H)$-invariant scalar product on V (cf. [3, Chapter X], for the last construction). Now, what is important is the case when H is not closed in G. Then the "homogeneous space G/H" exists only in the set-theoretical sense but not as a smooth manifold. Yet, Theorem A shows that such a homogeneous space still exists "locally" and it is represented by a "small manifold" M constructed above.

2. The main theorem

In this chapter we derive the coordinate expressions for the 5-dimensional examples mentioned in the introduction. Here, we shall use Theorem A and its Corollary as a theoretical foundation.

We shall also determine some interesting geometrical properties of these examples.

Our main result is the following

Theorem 1. *Let M denote the open strip $(-\frac{1}{2}\pi, \frac{1}{2}\pi)^2 \times \mathbb{R}^3$ of the Cartesian space $\mathbb{R}^5(\varrho, \tilde{\varrho}, \varphi, \tilde{\varphi}, t)$ and let, for $r \in \mathbb{R}$, a Riemannian metric g_r be defined on M by the formula*

$$g_r = (d\varrho)^2 + (d\tilde{\varrho})^2 + (\cos^2\varrho)\left[(d\varphi)^2 - \frac{2r}{\sqrt{r^2+1}}\,d\varphi\,dt\right] + (\cos^2\tilde{\varrho})\left[(d\tilde{\varphi})^2 + \frac{2}{\sqrt{r^2+1}}\,d\tilde{\varphi}\,dt\right] + (dt)^2.$$

Then the Riemannian manifold (M, g_r) is locally homogeneous for each $r \in \mathbb{R}$. Further, it is locally isometric to a globally homogeneous Riemannian space if, and only if, r is a rational number.

Proof. First, consider the group S^3 of unit quaternions $\mathbf{q} = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, $\sum(x_i)^2 = 1$, on which a local coordinate system (α, β, γ) (hyperspherical coordinates) is given by the expressions $x_1 = \cos\alpha\cos\beta\cos\gamma$, $x_2 = \cos\alpha\cos\beta\sin\gamma$, $x_3 = \cos\alpha\sin\beta$, $x_4 = \sin\alpha$, $(\alpha, \beta, \gamma) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)^2 \times (0, 2\pi)$. Using the standard multiplication in S^3 and the "Cartan's algorithm" (see [1]), one obtains easily a basis $\{X, Y, Z\}$ of left-invariant vector fields on S^3 given in the coordinate domain of (α, β, γ) by the formulas

$$\begin{aligned} X &= \cos\beta\cos\gamma\,\frac{\partial}{\partial\alpha} + (\sin\gamma + \tan\alpha\sin\beta\cos\gamma)\,\frac{\partial}{\partial\beta} \\ &\quad + \frac{1}{\cos\alpha\cos\beta}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma)\,\frac{\partial}{\partial\gamma}, \\ Y &= -\cos\beta\sin\gamma\,\frac{\partial}{\partial\alpha} + (\cos\gamma - \tan\alpha\sin\beta\sin\gamma)\,\frac{\partial}{\partial\beta} \\ &\quad + \frac{1}{\cos\alpha\cos\beta}(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma)\,\frac{\partial}{\partial\gamma}, \\ Z &= \sin\beta\,\frac{\partial}{\partial\alpha} - \tan\alpha\cos\beta\,\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\gamma}. \end{aligned} \tag{1}$$

Hence we check easily

$$[X, Y] = 2Z, \quad [Y, Z] = 2X, \quad [Z, X] = 2Y. \tag{2}$$

We now want to determine two independent solutions of the partial differential equation $Z(f) = 0$. First we see at once that

$$\sigma = \cos\alpha\cos\beta \tag{3}$$

is a solution which is independent of γ.

Using the Lie algebra structure (2) and the complexification of the function space, one obtains easily $Z\big(X\sigma + (Y\sigma)\mathbf{i}\big) = (-2\mathrm{i})\big(X\sigma + (Y\sigma)\mathbf{i}\big)$ and hence the (multivalued) function

$$f = \mathbf{i}\cdot\log\big(X\sigma + (Y\sigma)\mathbf{i}\big) - 2\gamma$$

is a complex solution of $Zf = 0$. In particular, using the standard formula

$$\log(x + y\mathbf{i}) = \log\sqrt{x^2 + y^2} + \arctan\left(\frac{y}{x}\right)\mathbf{i} + k\pi\mathbf{i} \tag{4}$$

(valid in the right open half-plane of $\mathbb{C}$), we see that $h = \arctan(Y\sigma/X\sigma) + 2\gamma$ is a real solution of $Zf = 0$, and so is the function $\tan h$. After an elementary calculation, $\tan h$ takes on the form of a *fraction*, whose numerator and denominator are independent solutions of $Zf = 0$.

More explicitly, the functions

$$\begin{aligned} u &= -\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma, \\ v &= \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \end{aligned} \tag{5}$$

are independent solutions of the equation $Zf = 0$ and, moreover,

$$u^2 + v^2 + \sigma^2 = 1. \tag{6}$$

Further, we have also useful relations

$$\begin{aligned} \sin\alpha &= u\cos\gamma + v\sin\gamma, \\ \cos\alpha\sin\beta &= -u\sin\gamma + v\cos\gamma. \end{aligned} \tag{7}$$

Now, introduce *new local coordinates* u, v and $w = \gamma$. Then first

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= (\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma)\frac{\partial}{\partial u} + (\cos\alpha\sin\gamma - \sin\alpha\sin\beta\cos\gamma)\frac{\partial}{\partial v}, \\ \frac{\partial}{\partial\beta} &= \sigma\left(-\sin\gamma\frac{\partial}{\partial u} + \cos\gamma\frac{\partial}{\partial v}\right), \\ \frac{\partial}{\partial\gamma} &= u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u} + \frac{\partial}{\partial w}. \end{aligned} \tag{8}$$

Now, using (7) and (3), one can rewrite (1) in the form

$$\begin{aligned} X &= a\frac{\partial}{\partial\alpha} + b\frac{\partial}{\partial\beta} + \frac{u\sin 2\gamma - v\cos 2\gamma}{\sigma}\frac{\partial}{\partial\gamma}, \\ Y &= c\frac{\partial}{\partial\alpha} + d\frac{\partial}{\partial\beta} + \frac{u\cos 2\gamma + v\sin 2\gamma}{\sigma}\frac{\partial}{\partial\gamma}, \\ Z &= \sin\beta\frac{\partial}{\partial\alpha} - \tan\alpha\cos\beta\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\gamma} \end{aligned} \tag{9}$$

where

$$\begin{aligned} a &= \cos\beta\cos\gamma, & b &= \sin\gamma + \tan\alpha\sin\beta\cos\gamma, \\ c &= -\cos\beta\sin\gamma, & d &= \cos\gamma - \tan\alpha\sin\beta\sin\gamma. \end{aligned} \tag{10}$$

Substituting (8) into (9) we get, after a lengthy but routine calculation

$$\sigma X = A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v} + E\frac{\partial}{\partial w}, \quad \sigma Y = C\frac{\partial}{\partial u} + D\frac{\partial}{\partial v} + F\frac{\partial}{\partial w}, \quad Z = \frac{\partial}{\partial w}, \tag{11}$$

where

$$
\begin{aligned}
A &= (1-u^2)\cos 2w - uv\sin 2w, & E &= u\sin 2w - v\cos 2w,\\
B &= (1-v^2)\sin 2w - uv\cos 2w, & F &= u\cos 2w + v\sin 2w,\\
C &= (u^2-1)\sin 2w - uv\cos 2w, & \sigma &= \sqrt{1-u^2-v^2} > 0,\\
D &= (1-v^2)\cos 2w + uv\sin 2w.
\end{aligned}
\tag{12}
$$

Using (6) we get the obvious identities

$$AF - CE = u\sigma^2, \quad BF - DE = v\sigma^2, \quad AD - BC = \sigma^2. \tag{13}$$

Hence and from (11) we get

$$\frac{\partial}{\partial u} = \frac{DX - BY + v\sigma Z}{\sigma}, \quad \frac{\partial}{\partial v} = \frac{-CX + AY - u\sigma Z}{\sigma} \tag{14}$$

and

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = \frac{1-v^2}{\sigma^2}, \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = \frac{uv}{\sigma^2}, \quad \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = \frac{1-u^2}{\sigma^2} \tag{15}$$

for a scalar product $\langle \cdot, \cdot \rangle$ with ortonormal basis $\{X, Y, Z\}$.

Next, we shall construct our locally homogeneous space on the Lie algebra level. (See [4, pp. 213–214] for the details—we are using a slightly different notation.)

Consider the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ written as $\operatorname{span}(X, Y, Z) \oplus \operatorname{span}(\tilde{X}, \tilde{Y}, \tilde{Z})$, where formula (2) is satisfied for both sets of vector fields. Introduce also hyperspherical coordinates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ on another 3-sphere $\tilde{S}^3$ and the variables $\tilde{u}, \tilde{v}, \tilde{w} = \tilde{\gamma}$ analogous to (5). Then the analogues with "tildes" of all formulas (1)–(15) are valid.

Suppose that a left-invariant metric $\tilde{g}$ is given on $G = S^3 \times \tilde{S}^3$ determined by the scalar product on $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ (identified with T_eG) for which $\{X, Y, Z, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ is an orthonormal basis. Further, consider the orthogonal decomposition

$$\mathfrak{g} = V + \mathfrak{h}_r, \tag{16}$$

where

$$V = \operatorname{span}(X, Y, \tilde{X}, \tilde{Y}, T), \qquad \mathfrak{h}_r = \operatorname{span}(\tilde{T}), \tag{17}$$

$$T = \frac{rZ - \tilde{Z}}{\sqrt{1+r^2}}, \qquad \tilde{T} = \frac{Z + r\tilde{Z}}{\sqrt{1+r^2}}. \tag{18}$$

The decomposition (16) is obviously $\operatorname{ad}(\mathfrak{h}_r)$-invariant and the operator $\operatorname{ad}(\tilde{T})$ is skew-symmetric on V.

In a neighborhood U of the identity $e \in G$ one can consider the new local coordinate system $(u, v, \tilde{u}, \tilde{v}, t = rw - \tilde{w}, \tilde{t} = w + r\tilde{w})$. Here we see that the corresponding coordinate vectors at the identity coincide with the vectors $X_e, Y_e, \tilde{X}_e, \tilde{Y}_e, T_e, \tilde{T}_e$, respectively. Indeed, at the identity element we have $\alpha = \beta = \gamma = 0, u = v = w = 0, \sigma = 1, A = D = 1, B = C = E = F = 0$, and similarly for the quantities with tildes. Moreover, we have in our coordinate neighborhood

$$\frac{\partial}{\partial t} = T = \frac{rZ - \tilde{Z}}{\sqrt{1+r^2}}. \tag{19}$$

Then we just use formula (11), its analogue, and formula (19).

We see that the conditions of Theorem A (and its Corollary) are satisfied. Hence the corresponding locally homogeneous Riemannian metric exists on a neighborhood M of the origin $o \in \mathbb{R}^5[u, v, \tilde{u}, \tilde{v}, t]$. Using the expressions (14), ($\widetilde{14}$), (19) and formulas (15), ($\widetilde{15}$), the metric components are calculated immediately and we obtain

$$\begin{aligned} g = &\frac{(1-v^2)\,du^2 + 2uv\,du\,dv + (1-u^2)\,dv^2}{\sigma^2} \\ &+ \frac{(1-\tilde{v}^2)\,d\tilde{u}^2 + 2\tilde{u}\tilde{v}\,d\tilde{u}\,d\tilde{v} + (1-\tilde{u}^2)\,d\tilde{v}^2}{\tilde{\sigma}^2} \\ &+ \frac{2}{\sqrt{r^2+1}}\big[r(v\,du - u\,dv) + (\tilde{u}\,d\tilde{v} - \tilde{v}\,d\tilde{u})\big]\,dt + (dt)^2. \end{aligned} \tag{20}$$

As the last step, we introduce a new system of local coordinates $(\varrho, \tilde{\varrho}, \varphi, \tilde{\varphi}, t)$, where

$$\begin{aligned} &u = \cos\varrho\cos\varphi, && \tilde{u} = \cos\tilde{\varrho}\cos\tilde{\varphi}, && \varrho, \tilde{\varrho} \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \\ &v = \cos\varrho\sin\varphi, && \tilde{v} = \cos\tilde{\varrho}\sin\tilde{\varphi}, && \varphi, \tilde{\varphi} \in (0, 2\pi). \end{aligned} \tag{21}$$

(Recall here that $u^2 + v^2 < 1$ by (6) and analogously $\tilde{u}^2 + \tilde{v}^2 < 1$.) Transforming the metric (20) into the new coordinates, we obtain exactly the formula from our Theorem 1. As concerns the definition domain of this metric, it can be obviously extended to the whole strip of $\mathbb{R}^5$ as indicated.

It follows from [4] that g_r comes from a globally homogeneous Riemannian space if and only if r is a rational number. (This is the case when the decomposition (16) produces a closed subgroup of G.) $\square$

We conclude with the following

Remarks. a) We see that the level hypersurfaces $t =$ const. of (M, g_r) are all locally isometric to the Riemannian product $S^2 \times \tilde{S}^2$ of standard unit 2-spheres. (For irrational r we have $S^2 \times \tilde{S}^2 = G/\text{closure}(H_r)$.)

b) It was proved in [7] that, for rational r, the corresponding homogeneous Riemannian space G/H_r is naturally reductive. For $r \in \mathbb{R}$ arbitrary, one can check easily that (M, g_r) admits at least so-called *naturally reductive homogeneous structure* (see [12], [13]). Then from [2] it follows that (M, g_r) is always a *space with volume-preserving local geodesic symmetries* (now called "D'Atri space"—cf. [5]).

c) The family (M, g_r) can be extended to a broader family of spaces $(M, g_{r,c,d})$ depending on additional two real parameters c, d. This can be done by assuming that $\{X, Y, Z, \tilde{X}, \tilde{Y}, \tilde{Z}\}$ is not an orthonormal basis but only an orthogonal basis satisfying

$$\|X\| = \|Y\| = c, \quad \|Z\| = c^2, \quad \|\tilde{X}\| = \|\tilde{Y}\| = d, \quad \|\tilde{Z}\| = d^2, \quad c, d > 0.$$

(Cf. [7, formula (34)], up to the notation.) The corresponding Riemannian metrics can be calculated easily and we shall not write them down explicitly. All the previous results remain valid for this extended family of Riemannian spaces.

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